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# SPARSE AITKEN-SCHWARZ WITH APPLICATION TO DARCY FLOW

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**Abstract.** This talk focuses on the acceleration of the Schwarz method by the Aitken's acceleration of the convergence technique with taken into account the special structure of the error operator. This allows an enhancement of the building of the low rank space approximating the solution at the gathered interfaces of all subdomains computed by singular values decomposition of the sequence of iterated solutions presents in the Aitken-Schwarz technique. The new method Sparse-Aitken Schwarz method builds low rank spaces associated to each subdomain's interfaces. Comparisons between Aitken-Schwarz and Sparse-Aitken Schwarz results obtained on a 3D Darcy flow application show the improvement by using the special structure of the propagation error operator.

## 1 INTRODUCTION

Schwarz domain decomposition methods is nowadays widely used to solve linear problems of the form $Ax = b$ because it is well suited for parallel computing. Indeed, it is based on the splitting of the global problem into subproblems. Artificial boundary conditions arise from the decomposition of the domain into subdomains. Then, the Schwarz method consists of the solution of subproblems, and the update of the artificial boundary conditions. In practice, this implies local communications between neighboring subdomains. The main drawback of Schwarz domain decomposition methods is the slow convergence, that depends on the nature of the problem, the geometry of the subdomains and the overlap. There exist several method for the acceleration of convergence, they are all based on the solution of small problem coupling all subdomains. The classical Aitken-Schwarz method [GTD02]($A{-}S$: the dash is to avoid the confusion with *Additive Schwarz*) consists in the approximation of the interface problem [TD09], and its solution via the Aitken's formula. In this talk, we take into account the sparsity structure of the of error propagation operator to build low rank approximations of the solution associated to individual artificial interface. The resulting method named the Sparse Aitken-Schwarz (SA-S) method shows better results of convergence and good parallel efficiency on 3D Darcy flow problem.

## 2 Numerical acceleration of the Schwarz method

A domain of $n$ unknowns is split in $N$ overlapping subdomains. The $i$th subdomain has $n_i$ if the overlap is included, or $\widetilde{n}_i$ if the overlap is excluded. In that case, $n = \Sigma_{i=0}^{N-1}\widetilde{n}_i$. Let $R_i \in \mathbb{R}^{n_i \times n}$ (respectively $\widetilde{R}_i \in \mathbb{R}^{n_i \times n}$) be the restriction operator of a global vector to the $i$th subdomain, including the overlap (respectively setting to 0 the components of the overlap). The additive Schwarz method with Dirichlet boundary conditions on the artificial boundary conditions can be written as the Richardson process:

$$x^{k+1} = x^k + M_{RAS}^{-1}\left(b - Ax^k\right) \tag{1}$$

with the matrix $M_{RAS}^{-1}$ is the Restricted Additive Schwarz (RAS) preconditioner :

$$M_{RAS}^{-1} = \sum_{i=0}^{N-1} \widetilde{R}_i^T \left(R_i A R_i^T\right)^{-1} R_i = \sum_{i=0}^{N-1} \widetilde{R}_i^T A_i^{-1} R_i. \tag{2}$$

If $x^\infty$ is the exact solution of the linear system $Ax = b$, and $x^k$ the solution at the $k$th Schwarz iteration, then subtracting two Schwarz iterations:

$$x^k - x^\infty = \left(I - M_{RAS}^{-1}A\right)\left(x^{k-1} - x^\infty\right) \tag{3}$$

which shows the purely linear convergence. This property still holds if we consider only the artificial interfaces. Let $R_\Gamma$ be the operator that restrict a vector to the artificial interface. The restriction of Eq.(1) to the interface is

$$\underbrace{R_\Gamma M_{RAS}^{-1} A R_\Gamma^T}_{I-P}\underbrace{R_\Gamma x}_{y} = \underbrace{R_\Gamma M_{RAS}^{-1} b}_{c}. \tag{4}$$

where the matrix $P := R_\Gamma\left(I - M_{RAS}^{-1}A\right)R_\Gamma^T$ is the error propagation operator since $e^{k+1} = Pe^k$ if $e^k = y^k - y^\infty$ is the error on the interface at iteration $k$. If the matrix $[y^{n_\Gamma} - y^{n_\Gamma - 1}, \cdots, y^1 - y^0]$ is not singular,the error propagation operator $P$ can be computed as: $P = [y^{n_\Gamma+1} - y^{n_\Gamma}, \ldots, y^2 - y^1]\,[y^{n_\Gamma} - y^{n_\Gamma-1}, \ldots, y^1 - y^0]^{-1}$. The solution $y^\infty$ can be computed as:

$$y^\infty = (I - P)^{-1}\left(y^{n_\Gamma+1} - Py^{n_\Gamma}\right). \tag{5}$$

In practice, it may not be possible to use the exact acceleration for 2D and 3D problems because it requires $n_\Gamma + 1$ Schwarz iterations, where $n_\Gamma$ is the number of unknowns on the artificial interface. Then, the acceleration is approximated in a low-dimensional space $U$ and an approximated propagation error operator $\widehat{P}$.

Eq.(5) can be written as (6) that requires the inversion of a matrix of size $l << n_\Gamma$.

$$y^\infty \approx \widetilde{y^\infty} = U\left(I - \widehat{P}\right)^{-1}\left(U^T y^q - \widehat{P} U^T y^{q-1}\right) \tag{6}$$

The Aitken's acceleration is given in Algorithm 1. The step 3 of this algorithm is the restriction of the Schwarz iterations to the interface, which is implemented as in Eq.(1).

**Algorithm 1** Approximated Aitken's Acceleration

**Require:** $y^0$ an initial guess
1: **repeat**
2:    **for** $i = 1 \dots q$ **do**
3:       $y^i \leftarrow Py^{i-1} + c$ //Schwarz iterations
4:    **end for**
5:    Compute a matrix $U$ with orthonormal columns such that $y^q$ and $y^{q-1} \in span(U)$
6:    Compute $\widehat{P}$ an approximation of $U^T P U$
7:    $y^0 \leftarrow U\left(I - \widehat{P}\right)^{-1}\left(U^T y^q - \widehat{P} U^T y^{q-1}\right)$
8: **until** convergence

Each iteration of the step 3 requires the solution of the local problems and the exchange of the artificial boundary conditions. It has been considered that the operator $P$ was approximated from $q+1$ successive Richardson's iterations. In step 5 the $U$ is computed with the SVD of the matrix $[y^0, \dots, y^q] = U\Sigma V^T$. So far, the approximation $\widehat{P}$ of $P$ given by $UPU^T$ was a full matrix but in fact the matrix $P$ can be very sparse. We propose a new methods, called sparse Aitken-Schwarz, to approximate the Aitken's acceleration that preserves the null blocks of the matrix $P$ corresponding to independent subdomains.

For two subdomains, let denote by $v_0^i$ and $v_1^i$ the solutions at the interface of the two subdomains at the $i$th iteration. We also denote by $R_{\Gamma_0}$ and $R_{\Gamma_1}$ the restriction operators to these two interfaces. Then, $v_0^i = R_{\Gamma_0} x^i$. The purely linear convergence of the Schwarz process can be written as $\begin{bmatrix} v_0^{i+1} - v_0^i \\ v_1^{i+1} - v_1^i \end{bmatrix} = R_\Gamma \left(I - M_{RAS}^{-1} A\right) R_\Gamma^T \begin{bmatrix} v_0^i - v_0^{i-1} \\ v_1^i - v_1^{i-1} \end{bmatrix}$ where the matrix $P = R_\Gamma (I - M_{RAS}^{-1} A) R_\Gamma^T$ can be decomposed $P = \begin{bmatrix} 0 & P_0 \\ P_1 & 0 \end{bmatrix}$.

Let $e_0^n = v_0^{i+1} - v_0^i$ and $e_1^i = v_1^{i+1} - v_1^i$ then:

$$P_0\left[e_1^0, \dots, e_1^{q-1}\right] = \left[e_0^1, \dots, e_0^q\right] \text{ and } P_1\left[e_0^0, \dots, e_0^{q-1}\right] = \left[e_1^1, \dots, e_1^q\right]. \tag{7}$$

In order to approximate the acceleration in low dimensional space, we compute independently the SVD of the trace of each interface. $U_i \Sigma_i V_i^T = [v_i^0, \dots, v_i^{q+1}]$ for $i = 0, 1$. Then

$$\left\{ \begin{array}{lcl} \widehat{P_0} & := & \left(U_0^T \left[e_0^1, \dots, e_0^q\right]\right)\left(U_1^T \left[e_1^0, \dots, e_1^{q-1}\right]\right)^{-1} \approx U_0^T P_0 U_1 \\ \widehat{P_1} & := & \left(U_1^T \left[e_1^1, \dots, e_1^q\right]\right)\left(U_0^T \left[e_0^0, \dots, e_0^{q-1}\right]\right)^{-1} \approx U_1^T P_1 U_0. \end{array} \right. \tag{8}$$

The approximation preserving the diagonal null blocs of the matrix $P$ is

$$P \approx \begin{bmatrix} 0 & U_0 \widehat{P_0} U_1^T \\ U_1 \widehat{P_1} U_0^T & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} U_0 & 0 \\ 0 & U_1 \end{bmatrix}}_{U} \times \underbrace{\begin{bmatrix} 0 & \widehat{P_0} \\ \widehat{P_1} & 0 \end{bmatrix}}_{\widehat{P}} \times \underbrace{\begin{bmatrix} U_0 & 0 \\ 0 & U_1 \end{bmatrix}^T}_{U^T}. \tag{9}$$

The Aitken's acceleration is computed after $q$ iterations, the dimension of the Krylov subspace is $q+1$. The matrix $U$ has $l \leq 2q+2$ columns linearly independent, spanning a subspace larger than the Krylov subspace if $l > q+1$. The Aitken's acceleration in which each blocs of $P$ has been approximated independently is more suitable for parallel computing because there is one SVD per interface instead of one global SVD.

## 3 Parallel performances

The groundwater flow in saturated media can be modelled using the Darcy's laws and the conservation of mass that gives Eq.(10), where $u$ is the hydraulic head $K(x,y,z)$ is the permeability field.

$$\left\{ \begin{array}{c} \nabla.\left(K(x,y,z)\ \nabla u\right) = 0 \text{ in } \Omega \\ u = \alpha, \text{ on } \Gamma_L, u = \beta, \text{ on } \Gamma_R, \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2) \end{array} \right. \tag{10}$$

The domain $\Omega$ is a parallelepiped, with two Dirichlet boundary conditions on the left $\Gamma_L$ and right $\Gamma_R$ wall, and homogeneous Neumann boundary conditions on the other walls. The problem is discretized using a standard 7 points stencil on a regular grid.

In order to test the weak scaling of our implementation, we set the size of one subdomain to $512 \times 512 \times 256$, and we increase the number of subdomains. Nine Schwarz iterations are computed before the acceleration, and one after. Table 1 shows the computational times and their repartition. The total number of Schwarz iterations is 10 for all considered

**Table 1**: Repartition of the computational time of $SA{-}S(9)$

| Subd. | Cores | Time (s) | Local solution | Aitken | Exchanges | Remaining |
|---|---|---|---|---|---|---|
| 2 | 512 | 752 | 99.444% | 0.123% | $1.54 \times 10^{-3}$% | 0.431% |
| 4 | 1024 | 811 | 99.051% | 0.186% | $2.86 \times 10^{-3}$% | 0.761% |
| 8 | 2048 | 828 | 98.548% | 0.168% | $4.35 \times 10^{-3}$% | 1.280% |
| 16 | 4096 | 817 | 98.063% | 0.208% | $5.00 \times 10^{-3}$% | 1.724% |

Dirichlet b.c. of 1.0 on the left and 10.0 on the right. 8 overlapping points between each subdomains. Subproblems solved by FGMRES preconditioned by Hypre with a relative tolerance of $10^{-12}$.

sizes of meshes. This means that the required tolerance $\|b - Ax\|_2 / \|b\|_2 < 10^{-5}$ has been reached after the first acceleration.

## REFERENCES

[GTD02] M. Garbey and D. Tromeur-Dervout. On some Aitken-like acceleration of the Schwarz method. *Internat. J. Numer. Methods Fluids*, 40(12):1493–1513, 2002. LMS Workshop on Domain Decomposition Methods in Fluid Mechanics (London, 2001).

[TD09] D. Tromeur-Dervout. Meshfree Adaptative Aitken-Schwarz Domain Decomposition with application to Darcy Flow. In Topping, BHV and Ivanyi, P, editor, *Parallel, Distributed and Grid Computing for Engineering*, volume 21 of *Computational Science Engineering and Technology Series*, pages 217–250. Saxe-Coburg Publications, Stirlingshire, UK, 2009.